

TRANSFERENCE OF FRACTIONAL LAPLACIAN REGULARITY

LUZ RONCAL AND PABLO RAÚL STINGA

ABSTRACT. In this note we show how to obtain regularity estimates for the fractional Laplacian on the multidimensional torus \mathbb{T}^n from the fractional Laplacian on \mathbb{R}^n . Though at first glance this may seem quite natural, it must be carefully precised. A reason for that is the simple fact that L^2 functions on the torus can not be identified with L^2 functions on \mathbb{R}^n . The transference is achieved through a formula that holds in the distributional sense. Such an identity allows us to transfer Harnack inequalities, to relate the extension problems, and to obtain pointwise formulas and Hölder regularity estimates.

1. THE TRANSFERENCE FORMULA

For $0 < \sigma < 1$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$, the fractional Laplacian of order σ in \mathbb{R}^n is defined using the Fourier transform as

$$(-\Delta_{\mathbb{R}^n})^\sigma u(x) = \int_{\mathbb{R}^n} |\xi|^{2\sigma} \widehat{u}(\xi) e^{ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^n.$$

Similarly, the fractional Laplacian on $\mathbb{T}^n \equiv \mathbb{R}^n / (2\pi\mathbb{Z}^n)$ is defined via the multiple Fourier series

$$(-\Delta_{\mathbb{T}^n})^\sigma v(z) = \sum_{k \in \mathbb{Z}^n} |k|^{2\sigma} c_k(v) e^{ik \cdot z},$$

where $c_k(v)$ is the Fourier coefficient of $v : \mathbb{T}^n \rightarrow \mathbb{R}$. In our notation, the point $(e^{iz_1}, \dots, e^{iz_n}) \in \mathbb{T}^n$ is uniquely identified with $z = (z_1, \dots, z_n) \in Q_n := (-\pi, \pi]^n$, so $v(z)$ in fact means $v(e^{iz_1}, \dots, e^{iz_n})$. In order to avoid a rather cumbersome notation, we will just write $z \in \mathbb{T}^n$.

It is clear that the fractional Laplacian on \mathbb{R}^n does not preserve the Schwartz class \mathcal{S} . Instead,

$$(-\Delta_{\mathbb{R}^n})^\sigma : \mathcal{S} \rightarrow \mathcal{S}_\sigma := \{\varphi \in C^\infty(\mathbb{R}^n) : (1 + |x|^2)^{\frac{n+2\sigma}{2}} D^k \varphi(x) \in L^\infty(\mathbb{R}^n), k \in \mathbb{N}_0\},$$

see [5, pp. 72–73]. Observe that $\mathcal{S} \subset \mathcal{S}_\sigma$. Then the symmetry of the fractional Laplacian allows us to define $(-\Delta_{\mathbb{R}^n})^\sigma$ for u in the dual space \mathcal{S}'_σ . For locally integrable functions u in \mathcal{S}'_σ we let

$$\langle (-\Delta_{\mathbb{R}^n})^\sigma u, \varphi \rangle_{\mathcal{S}_\sigma} := \int_{\mathbb{R}^n} u(x) (-\Delta_{\mathbb{R}^n})^\sigma \varphi(x) dx, \quad \varphi \in \mathcal{S}.$$

Certainly, the integral above is absolutely convergent when (see also [5])

$$u \in L_\sigma := L^1(\mathbb{R}^n, (1 + |x|^2)^{-\frac{n+2\sigma}{2}} dx).$$

The situation with the fractional Laplacian on the torus is different than the \mathbb{R}^n case. We first notice that $(-\Delta_{\mathbb{T}^n})^\sigma$ preserves the class of smooth functions on \mathbb{T}^n . By symmetry we are able to define this operator for any function v that is a periodic distribution on \mathbb{T}^n . Indeed, we let

$$\langle (-\Delta_{\mathbb{T}^n})^\sigma v, \phi \rangle_{C^\infty(\mathbb{T}^n)} := \int_{\mathbb{T}^n} v(z) (-\Delta_{\mathbb{T}^n})^\sigma \phi(z) dz, \quad \phi \in C^\infty(\mathbb{T}^n).$$

To relate both fractional Laplacians we define two operators. For a function v on \mathbb{T}^n we define its *repetition* $Rv : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$(Rv)(x) = \sum_{k \in \mathbb{Z}^n} v(x - 2\pi k) \chi_{Q_n}(x - 2\pi k), \quad x \in \mathbb{R}^n.$$

2010 *Mathematics Subject Classification.* Primary: 35R11, 35B65. Secondary: 26A33, 47G20.

Key words and phrases. Fractional Laplacian, transference, Harnack inequality, extension problem, Hölder regularity.

The first author was partially supported by grant MTM2012-36732-C03-02 from Spanish Government. The second author was partially supported by MTM2011-28149-C02-01 from Spanish Government.

This is nothing but the Q_n -periodic function on \mathbb{R}^n that coincides with v on \mathbb{T}^n . Here \mathbb{T}^n is identified with Q_n as explained above. For a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ we define its *periodization* as the function $p_\Sigma u : \mathbb{T}^n \rightarrow \mathbb{R}$ given (formally) by

$$(1.1) \quad (p_\Sigma u)(z) = \sum_{k \in \mathbb{Z}^n} u(z + 2\pi k), \quad z \in \mathbb{T}^n.$$

Theorem A (Transference formula). *Let v be a function on the torus such that*

$$(1.2) \quad \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |c_k(v)| \frac{e^{-|k|^2}}{|k|} < \infty.$$

Then its repetition Rv is a function in L_σ which defines a distribution in \mathcal{S}'_σ and such that

$$(1.3) \quad \int_{\mathbb{R}^n} (Rv)(-\Delta_{\mathbb{R}^n})^\sigma \varphi \, dx = \int_{\mathbb{T}^n} v(-\Delta_{\mathbb{T}^n})^\sigma (p_\Sigma \varphi) \, dz, \quad \varphi \in \mathcal{S}.$$

In other words, when evaluated in periodizations of Schwartz functions, the periodic distribution $(-\Delta_{\mathbb{T}^n})^\sigma v$ coincides with the distributional fractional Laplacian on \mathbb{R}^n of its repetition Rv .

Proof. We first check that $Rv \in L_\sigma$. Let us compute

$$(1.4) \quad \begin{aligned} \int_{\mathbb{R}^n} \frac{|(Rv)(x)|}{(1+|x|^2)^{\frac{n+2\sigma}{2}}} \, dx &= \sum_{k \in \mathbb{Z}^n} \int_{Q_n} \frac{|(Rv)(x+2k\pi)|}{(1+|x+2k\pi|^2)^{\frac{n+2\sigma}{2}}} \, dx \\ &= \int_{\mathbb{T}^n} |v(z)| p_\Sigma((1+|\cdot|^2)^{-\frac{n+2\sigma}{2}})(z) \, dz. \end{aligned}$$

Since $(1+|\cdot|^2)^{-\frac{n+2\sigma}{2}}$ is integrable, then $[p_\Sigma(1+|\cdot|^2)^{-\frac{n+2\sigma}{2}}]$ is integrable (see [6, Chapter VII]). Its Fourier coefficient can be computed as follows:

$$\begin{aligned} \mathcal{F}[(1+|\cdot|^2)^{-\frac{n+2\sigma}{2}}](k) &= \mathcal{F}^{-1}(\mathcal{F}(I - \Delta_{\mathbb{R}^n})^{-\frac{n+2\sigma}{2}})(k) \\ &= \frac{1}{\Gamma(\frac{n+2\sigma}{2})} \int_0^\infty e^{-t} \frac{e^{-|k|^2/(4t)}}{(4\pi t)^{n/2}} \frac{dt}{t^{1-\frac{n+2\sigma}{2}}} \\ &= \frac{|k|^{2\sigma}}{(4\pi)^{n/2} 4^\sigma \Gamma(\frac{n+2\sigma}{2})} \int_0^\infty e^{-|k|^2/(4r)} e^{-r} \frac{dr}{r^{1+\sigma}} = c_{n,\sigma} K_\sigma(|k|^2). \end{aligned}$$

Here $K_\sigma(z)$ is the modified Bessel function of the third kind (see [3, p. 119]). A well known asymptotic formula gives that $K_\sigma(|k|^2) \sim |k|^{-1} e^{-|k|^2}$, as $|k| \rightarrow \infty$. Hence, by Parseval's identity on \mathbb{T}^n and the hypothesis, from (1.4) we get

$$\int_{\mathbb{R}^n} \frac{|(Rv)(x)|}{(1+|x|^2)^{\frac{n+2\sigma}{2}}} \, dx = c_{n,\sigma} \sum_{k \in \mathbb{Z}^n} |c_k(v)| K_\sigma(|k|^2) \leq C_{n,\sigma} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |c_k(v)| \frac{e^{-|k|^2}}{|k|} < \infty.$$

Thus, $Rv \in L_\sigma$ and the left hand side of (1.3) is absolutely convergent.

Again, $p_\Sigma \varphi$ is integrable on \mathbb{T}^n and $c_k(p_\Sigma \varphi) = \hat{\varphi}(k)$, for each $k \in \mathbb{Z}^n$. Moreover, since φ and $\hat{\varphi}$ decay at infinity as $|x|^{-n-\delta}$, $\delta > 0$, we have

$$(1.5) \quad (p_\Sigma \varphi)(z) = \sum_{k \in \mathbb{Z}^n} \hat{\varphi}(k) e^{ik \cdot z},$$

where the series converges absolutely, see [6, Chapter VII]. From here, using the properties of the Fourier transform, it readily follows that $p_\Sigma \varphi$ is a smooth function on the torus. Hence $(-\Delta_{\mathbb{T}^n})^\sigma (p_\Sigma \varphi)$ is smooth too and the right hand side of (1.3) is absolutely convergent.

Before proving (1.3) we compute the periodization of $(-\Delta_{\mathbb{R}^n})^\sigma \varphi$. Since φ is in the Schwartz class, both $(-\Delta_{\mathbb{R}^n})^\sigma \varphi$ and its Fourier transform decay as $|x|^{-(n+2\sigma)}$ at infinity. Therefore, by (1.5),

$$\begin{aligned} [p_\Sigma(-\Delta_{\mathbb{R}^n})^\sigma \varphi](z) &= \sum_{k \in \mathbb{Z}^n} \widehat{(-\Delta_{\mathbb{R}^n})^\sigma \varphi}(k) e^{ik \cdot z} = \sum_{k \in \mathbb{Z}^n} |k|^{2\sigma} \widehat{\varphi}(k) e^{ik \cdot z} \\ &= \sum_{k \in \mathbb{Z}^n} |k|^{2\sigma} c_k(p_\Sigma \varphi) e^{ik \cdot z} = (-\Delta_{\mathbb{T}^n})^\sigma (p_\Sigma \varphi)(z), \end{aligned}$$

for each $z \in \mathbb{T}^n$. With this, we readily obtain

$$\begin{aligned} \int_{\mathbb{R}^n} (Rv)(-\Delta_{\mathbb{R}^n})^\sigma \varphi \, dx &= \int_{\mathbb{R}^n} \left[\sum_{k \in \mathbb{Z}^n} v(x - 2\pi k) \chi_{Q_n}(x - 2\pi k) \right] (-\Delta_{\mathbb{R}^n})^\sigma \varphi(x) \, dx \\ &= \sum_{k \in \mathbb{Z}^n} \int_{Q_n + 2\pi k} v(x - 2\pi k) (-\Delta_{\mathbb{R}^n})^\sigma \varphi(x) \, dx \\ &= \sum_{k \in \mathbb{Z}^n} \int_{Q_n} v(z) (-\Delta_{\mathbb{R}^n})^\sigma \varphi(z + 2\pi k) \, dz \\ &= \int_{\mathbb{T}^n} v(z) [p_\Sigma(-\Delta_{\mathbb{R}^n})^\sigma \varphi](z) \, dz = \int_{\mathbb{T}^n} v(-\Delta_{\mathbb{T}^n})^\sigma (p_\Sigma \varphi) \, dz. \end{aligned}$$

Notice that the integration on the torus with respect to the Haar measure is just the integration over Q_n with respect to the Lebesgue measure, so the previous to last equality is true. \square

Remark 1.1. Formula (1.3) is certainly valid for functions $v \in L^p(\mathbb{T}^n)$, $1 \leq p \leq \infty$. Indeed, $v \in L^1(\mathbb{T}^n)$ and, by the Riemann–Lebesgue Lemma, $c_k(v) \rightarrow 0$ as $|k| \rightarrow \infty$, thus (1.2) holds. Observe that condition (1.2) also holds whenever $\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{2\sigma} |c_k(v)|^2 < \infty$, that is, when v is in the Sobolev space $H^\sigma = \text{Dom}((-\Delta_{\mathbb{T}^n})^{\sigma/2})$.

2. APPLICATIONS

2.1. Harnack inequalities. Interior and boundary Harnack estimates for the fractional Laplacian on the torus now follow from the transference formula in Theorem A.

Theorem 2.1 (Interior Harnack inequality). *Let $\mathcal{O} \subseteq \mathbb{T}^n$ be an open set. For any compact subset $\mathcal{K} \subset \mathcal{O}$, there exists a constant $C > 0$, that depends only on n, σ and \mathcal{K} , such that*

$$\sup_{\mathcal{K}} v \leq C \inf_{\mathcal{K}} v,$$

for all solutions $v \in \text{Dom}((-\Delta_{\mathbb{T}^n})^\sigma)$ to

$$\begin{cases} (-\Delta_{\mathbb{T}^n})^\sigma v = 0, & \text{in } \mathcal{O}, \\ v \geq 0, & \text{on } \mathbb{T}^n. \end{cases}$$

Proof. For v as in the hypothesis, its repetition Rv is a nonnegative function on \mathbb{R}^n which belongs to L_σ . We can identify \mathcal{O} with an open subset $\tilde{\mathcal{O}} \subset Q_n$. Take any smooth function φ with compact support in $\tilde{\mathcal{O}}$. Then $p_\Sigma \varphi$ is a smooth function on the torus supported in \mathcal{O} . Now Theorem A gives that $\langle (-\Delta_{\mathbb{R}^n})^\sigma (Rv), \varphi \rangle_{S_\sigma} = \langle (-\Delta_{\mathbb{T}^n})^\sigma v, p_\Sigma \varphi \rangle_{C^\infty(\mathbb{T}^n)} = 0$. Hence Rv is a nonnegative solution to $(-\Delta_{\mathbb{R}^n})^\sigma (Rv) = 0$ in $\tilde{\mathcal{O}}$. Then Rv satisfies Harnack inequality (see [1, Theorem 5.1]), and so does v . \square

Theorem 2.2 (Boundary Harnack inequality). *Let $0 < \sigma < 1$ and $v_1, v_2 \in \text{Dom}((-\Delta_{\mathbb{T}^n})^\sigma)$ be two nonnegative functions on \mathbb{T}^n . Suppose that $(-\Delta_{\mathbb{T}^n})^\sigma v_j = 0$ in \mathcal{O} , for some open set $\mathcal{O} \subseteq \mathbb{T}^n$. Let $z_0 \in \partial \mathcal{O}$ and assume that $v_j = 0$ for all $z \in B_r(z_0) \cap \mathcal{O}^c$, for some sufficiently small $r > 0$. Assume also that $\partial \mathcal{O} \cap B_r(z_0)$ is a Lipschitz graph in the direction of z_1 . Then, there is a constant C depending only on \mathcal{O}, z_0, r, n and σ , but not on v_1 or v_2 , such that*

$$\sup_{\mathcal{O} \cap B_{r/2}(z_0)} \left(\frac{v_1}{v_2} \right) \leq C \inf_{\mathcal{O} \cap B_{r/2}(z_0)} \left(\frac{v_1}{v_2} \right).$$

Moreover, v_1/v_2 is α -Hölder continuous in $\overline{\mathcal{O} \cap B_{r/2}(z_0)}$, for some universal $0 < \alpha < 1$.

Proof. Again we have that Rv_i is in L_σ . We identify \mathcal{O} with an open subset $\tilde{\mathcal{O}} \subset Q_n$. Let us also identify $z_0 \in \partial\mathcal{O}$ with $x_0 \in \partial\tilde{\mathcal{O}}$. Then the corresponding boundary portion $\partial\tilde{\mathcal{O}} \cap B_r(x_0)$ is a Lipschitz graph in the x_1 -direction. Using the same argument as in the proof of Theorem 2.1, it follows that Rv_i are nonnegative solutions to $(-\Delta_{\mathbb{R}^n})^\sigma(Rv_i) = 0$ in $\tilde{\mathcal{O}}$, and $Rv_i = 0$ in $B_r(x_0) \cap \tilde{\mathcal{O}}^c$. Therefore, the boundary Harnack inequality holds for Rv_i (see [1, Theorem 5.3]), and so does for v_i by restricting Rv_i to \mathcal{O} . The Hölder continuity of v_1/v_2 follows from the Hölder continuity for $(Rv_1)/(Rv_2)$. \square

2.2. Extension problem. It is known that the Caffarelli–Silvestre extension problem characterization is valid also for the fractional Laplacian on the torus, see [7, 8], also [4, 2]. Here we can derive it directly from the Caffarelli–Silvestre result of \mathbb{R}^n in [1] with the explicit constants computed in [8]. In the proof we are going to need the following simple result.

Lemma 2.3. *Let ϕ be a smooth function on \mathbb{T}^n . Then there exists a smooth function φ with compact support on \mathbb{R}^n such that*

$$\phi(z) = p_\Sigma \varphi(z), \quad \text{for } z \in \mathbb{T}^n.$$

Proof. It is easy to see that there exists a smooth function ψ with compact support on \mathbb{R}^n such that $\sum_{k \in \mathbb{Z}^n} \psi(x + 2\pi k) \equiv 1$, for all $x \in \mathbb{R}^n$. Indeed, ψ can be constructed as the convolution of the characteristic function of Q_n with a smooth bump function that has integral 1. Set $\varphi(x) = \psi(x)(R\phi)(x)$. Clearly, φ is smooth (see the proof of Theorem A) and has compact support. Moreover,

$$\begin{aligned} (p_\Sigma \varphi)(z) &= \sum_{k \in \mathbb{Z}^n} \psi(z + 2\pi k)(R\phi)(z + 2\pi k) \\ &= \sum_{k \in \mathbb{Z}^n} \psi(z + 2\pi k) \sum_{j \in \mathbb{Z}^n} \phi(z + 2\pi k + 2\pi j) \chi_{Q_n}(z + 2\pi k + 2\pi j) \\ &= \phi(z) \sum_{k \in \mathbb{Z}^n} \psi(z + 2\pi k) = \phi(z), \quad z \in \mathbb{T}^n. \end{aligned}$$

\square

Theorem 2.4 (Extension problem). *Let $v \in \text{Dom}((-\Delta_{\mathbb{T}^n})^\sigma)$. Let $V = V(z, y)$ be the solution to the boundary value problem*

$$(2.1) \quad \begin{cases} \Delta_{\mathbb{T}^n} V + \frac{1-2\sigma}{y} V_y + V_{yy} = 0, & \text{in } \mathbb{T}^n \times (0, \infty), \\ V(z, 0) = v(z), & \text{on } \mathbb{T}^n. \end{cases}$$

Then, for $c_\sigma = \frac{\Gamma(1-\sigma)}{4^{\sigma-1/2}\Gamma(\sigma)} > 0$, we have that

$$(2.2) \quad - \lim_{y \rightarrow 0^+} y^{1-2\sigma} V_y(z, y) = c_\sigma (-\Delta_{\mathbb{T}^n})^\sigma v(z), \quad z \in \mathbb{T}^n.$$

Proof. Consider $u = Rv \in L_\sigma$. Let U be the solution to the extension problem for u :

$$\begin{cases} \Delta_{\mathbb{R}^n} U + \frac{1-2\sigma}{y} U_y + U_{yy} = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ U(x, 0) = u(x), & \text{on } \mathbb{R}^n. \end{cases}$$

From [1] we know that $U(x, y) = P_y^\sigma * u(x)$, for a suitable Poisson kernel $P_y^\sigma(x)$. Using this Poisson formula and analogous to the proof of Theorem A, it can be checked that $U(z, y) = (v * (p_\Sigma P_y^\sigma))(z)$, $z \in \mathbb{T}^n$, where the convolution is performed on \mathbb{T}^n . Then $U(z, y)$ is a solution to (2.1). By uniqueness,

it follows that $V(\cdot, y) = v * (p_\Sigma P_y^\sigma)$, for each $y > 0$. Moreover, by Theorem A and the Caffarelli–Silvestre extension result for the fractional Laplacian on \mathbb{R}^n in [1],

$$\begin{aligned} c_\sigma \int_{\mathbb{T}^n} v(-\Delta_{\mathbb{T}^n})^\sigma (p_\Sigma \varphi) dz &= c_\sigma \int_{\mathbb{R}^n} u(-\Delta_{\mathbb{R}^n})^\sigma \varphi dx = - \lim_{y \rightarrow 0^+} \int_{\mathbb{R}^n} y^{1-2\sigma} U_y(x, y) \varphi(x) dx \\ &= - \lim_{y \rightarrow 0^+} \sum_{k \in \mathbb{Z}^n} \int_{Q_n} y^{1-2\sigma} U_y(z + 2\pi k, y) \varphi(z + 2\pi k) dz \\ &= - \lim_{y \rightarrow 0^+} \int_{Q_n} y^{1-2\sigma} U_y(z, y) (p_\Sigma \varphi)(z) dz \\ &= - \lim_{y \rightarrow 0^+} \int_{\mathbb{T}^n} y^{1-2\sigma} V_y(z, y) (p_\Sigma \varphi)(z) dz. \end{aligned}$$

Now (2.2) follows because any smooth function ϕ on the torus can be expressed as $p_\Sigma \varphi$, for some $\varphi \in \mathcal{S}$, see Lemma 2.3. \square

2.3. Pointwise formula. Let $0 < \alpha \leq 1$ and $k \in \mathbb{N}_0$. A continuous real function v defined on \mathbb{T}^n belongs to the Hölder space $C^{k, \alpha}(\mathbb{T}^n)$, if $v \in C^k(\mathbb{T}^n)$ and

$$[D^\gamma v]_{C^\alpha(\mathbb{T}^n)} := \sup_{\substack{x, y \in \mathbb{T}^n \\ x \neq y}} \frac{|D^\gamma v(x) - D^\gamma v(y)|}{(x, y)^\alpha} < \infty,$$

for each multi-index $\gamma \in \mathbb{N}_0^n$ such that $|\gamma| = k$. Here (x, y) is the geodesic distance from x to y on \mathbb{T}^n . We define the norm in the spaces $C^{k, \alpha}(\mathbb{T}^n)$ as usual.

Theorem 2.5 (Pointwise formula). *Let $v \in C^{0, 2\sigma+\varepsilon}(\mathbb{T}^n)$ if $0 < \sigma < 1/2$ (or $v \in C^{1, 2\sigma+\varepsilon-1}(\mathbb{T}^n)$ if $1/2 \leq \sigma < 1$). Then $(-\Delta_{\mathbb{T}^n})^\sigma v$ coincides with the continuous function on \mathbb{T}^n given by*

$$\begin{aligned} (-\Delta_{\mathbb{T}^n})^\sigma v(x) &= \text{P. V.} \int_{\mathbb{T}^n} (v(x) - v(z)) K_\sigma(x - z) dz \\ &= \lim_{\delta \rightarrow 0^+} \int_{|x-z| > \delta, z \in \mathbb{T}^n} (v(x) - v(z)) K_\sigma(x - z) dz, \quad x \in \mathbb{T}^n, \end{aligned}$$

where, for $x \in \mathbb{T}^n$, $x \neq 0$,

$$K_\sigma(x) = \frac{2^\sigma \Gamma(\frac{n+\sigma}{2})}{|\Gamma(-\sigma/2)| \pi^{n/2}} \sum_{k \in \mathbb{Z}^n} \frac{1}{|x + 2\pi k|^{n+2\sigma}}.$$

In the case $0 < \sigma < 1/2$ the integral above is in fact absolutely convergent.

One may think that K_σ is just the periodization of the kernel of the fractional Laplacian on \mathbb{R}^n . In fact, formally, $K_\sigma(x) = c_{n, \sigma} p_\Sigma(|x|^{-(n+2\sigma)})$. But, since $|x|^{-(n+2\sigma)}$ is not integrable on \mathbb{R}^n , this formal identity makes no sense.

Proof of Theorem 2.5. Notice that $K_\sigma(x)$ is well defined for $x \neq 0$. Indeed, if $k \neq 0$, then for $x \in \mathbb{T}^n$ we have $|\pi k| \leq c_n |x - 2k\pi|$, so

$$0 \leq K_\sigma(x) \leq C_{n, \sigma} \left[\frac{1}{|x|^{n+2\sigma}} + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|\pi k|^{n+2\sigma}} \right], \quad x \neq 0,$$

and the series is absolutely convergent. We have to prove that

$$(2.3) \quad \langle (-\Delta_{\mathbb{T}^n})^\sigma v, \phi \rangle_{C^\infty(\mathbb{T}^n)} = \int_{\mathbb{T}^n} h(x) \phi(x) dx, \quad \text{for any } \phi \in C^\infty(\mathbb{T}^n),$$

where the continuous function h is given by

$$h(x) = \text{P. V.} \int_{\mathbb{T}^n} (v(x) - v(z)) K_\sigma(x - z) dz.$$

Let $u = Rv$. Then u is bounded and it belongs to $C^{0,2\sigma+\varepsilon}(\mathbb{R}^n)$ (or to $C^{1,2\sigma+\varepsilon-1}(\mathbb{R}^n)$), so $(-\Delta_{\mathbb{R}^n})^\sigma u$ is a continuous function on \mathbb{R}^n (see [5, Proposition 2.4]) and

$$\begin{aligned}
 (-\Delta_{\mathbb{R}^n})^\sigma u(x) &= c_{n,\sigma} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy \\
 &= c_{n,\sigma} \text{P.V.} \sum_{k \in \mathbb{Z}^n} \int_{Q_n} \frac{u(x) - u(z - 2\pi k)}{|x - z + 2\pi k|^{n+2\sigma}} dz \\
 &= c_{n,\sigma} \text{P.V.} \sum_{k \in \mathbb{Z}^n} \int_{Q_n} \frac{v(x) - v(z)}{|x - z + 2\pi k|^{n+2\sigma}} dz \\
 &= c_{n,\sigma} \text{P.V.} \int_{\mathbb{T}^n} (v(x) - v(z)) K_\sigma(x - z) dz = h(x).
 \end{aligned}
 \tag{2.4}$$

With this we conclude that h is a continuous function on \mathbb{T}^n . Observe that $(-\Delta_{\mathbb{R}^n})^\sigma u$ is a Q_n -periodic function. To establish (2.3), let ϕ be any smooth function on the torus. By Lemma 2.3, there exists $\varphi \in \mathcal{S}$ such that $\phi(z) = p_\Sigma \varphi(z)$, $z \in \mathbb{T}^n$. Then, by Theorem A and (2.4),

$$\begin{aligned}
 \langle (-\Delta_{\mathbb{T}^n})^\sigma v, \phi \rangle_{C^\infty(\mathbb{T}^n)} &= \langle (-\Delta_{\mathbb{T}^n})^\sigma v, p_\Sigma \varphi \rangle_{C^\infty(\mathbb{T}^n)} = \langle (-\Delta_{\mathbb{R}^n})^\sigma u, \varphi \rangle_{\mathcal{S}_\sigma} \\
 &= \int_{\mathbb{R}^n} (-\Delta_{\mathbb{R}^n})^\sigma u(x) \varphi(x) dx \\
 &= \sum_{k \in \mathbb{Z}^n} \int_{Q_n} (-\Delta_{\mathbb{R}^n})^\sigma u(x + 2\pi k) \varphi(x + 2\pi k) dx \\
 &= \int_{Q_n} (-\Delta_{\mathbb{R}^n})^\sigma u(x) (p_\Sigma \varphi)(x) dx = \int_{\mathbb{T}^n} h(x) \phi(x) dx.
 \end{aligned}$$

□

2.4. Hölder regularity. Hölder estimates follow directly from our transference formula and the known results for the fractional Laplacian on \mathbb{R}^n .

Theorem 2.6 (Hölder estimates). *Take $\alpha \in (0, 1]$.*

- (1) *Let $v \in C^{0,\alpha}(\mathbb{T}^n)$ and $0 < 2\sigma < \alpha$. Then $(-\Delta_{\mathbb{T}^n})^\sigma v \in C^{0,\alpha-2\sigma}(\mathbb{T}^n)$ and*

$$\|(-\Delta_{\mathbb{T}^n})^\sigma v\|_{C^{0,\alpha-2\sigma}(\mathbb{T}^n)} \leq C \|v\|_{C^{0,\alpha}(\mathbb{T}^n)}.$$

- (2) *Let $v \in C^{1,\alpha}(\mathbb{T}^n)$ and $0 < 2\sigma < \alpha$. Then $(-\Delta_{\mathbb{T}^n})^\sigma v \in C^{1,\alpha-2\sigma}(\mathbb{T}^n)$ and*

$$\|(-\Delta_{\mathbb{T}^n})^\sigma v\|_{C^{1,\alpha-2\sigma}(\mathbb{T}^n)} \leq C \|v\|_{C^{1,\alpha}(\mathbb{T}^n)}.$$

- (3) *Let $v \in C^{1,\alpha}(\mathbb{T}^n)$ and $2\sigma \geq \alpha$, with $\alpha - 2\sigma + 1 \neq 0$. Then $(-\Delta_{\mathbb{T}^n})^\sigma v \in C^{0,\alpha-2\sigma+1}(\mathbb{T}^n)$ and*

$$\|(-\Delta_{\mathbb{T}^n})^\sigma v\|_{C^{0,\alpha-2\sigma+1}(\mathbb{T}^n)} \leq C \|v\|_{C^{1,\alpha}(\mathbb{T}^n)}.$$

- (4) *Let $v \in C^{k,\alpha}(\mathbb{T}^n)$ and assume that $k + \alpha - 2\sigma$ is not an integer. Then $(-\Delta_{\mathbb{T}^n})^\sigma v \in C^{l,\beta}(\mathbb{T}^n)$, where l is the integer part of $k + \alpha - 2\sigma$ and $\beta = k + \alpha - 2\sigma - l$, and*

$$\|(-\Delta_{\mathbb{T}^n})^\sigma v\|_{C^{l,\beta}(\mathbb{T}^n)} \leq C \|v\|_{C^{k,\alpha}(\mathbb{T}^n)}.$$

Proof. For (1), by Theorem 2.5 and [5, Proposition 2.5] we readily get,

$$\|(-\Delta_{\mathbb{T}^n})^\sigma v\|_{C^{0,\alpha-2\sigma}(\mathbb{T}^n)} = \|(-\Delta_{\mathbb{T}^n})^\sigma (Rv)\|_{C^{0,\alpha-2\sigma}(\mathbb{R}^n)} \leq C \|Rv\|_{C^{0,\alpha}(\mathbb{R}^n)} = C \|v\|_{C^{0,\alpha}(\mathbb{T}^n)}.$$

Parts (2), (3) and (4) follow analogously by using Theorem A, Theorem 2.5 and the known results for \mathbb{R}^n [5, Proposition 2.6, Proposition 2.7]. □

Acknowledgement. We thank Luis Caffarelli and José L. Torrea for delightful and pleasant discussions about the results of this work.

REFERENCES

- [1] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245–1260.
- [2] J. E. Galé, P. J. Miana and P. R. Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* **13** (2013), 343–368.
- [3] N. N. Lebedev, *Special Functions and Its Applications*, Dover, New York, 1972.
- [4] L. Roncal and P. R. Stinga, Fractional Laplacian on the torus, arXiv:1209.6104v2, *preprint* (2012), 25pp.
- [5] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), 67–112.
- [6] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series **32**, Princeton Univ. Press, Princeton, New Jersey, 1971.
- [7] P. R. Stinga, Fractional powers of second order partial differential operators: extension problem and regularity theory, PhD thesis, Universidad Autónoma de Madrid (2010).
- [8] P. R. Stinga and J. L. Torrea, Extension problem and Harnack’s inequality for some fractional operators, *Comm. Partial Differential Equations* **35** (2010), 2092–2122.

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, 26004 LOGROÑO, SPAIN
E-mail address: luz.roncal@unirioja.es

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION C1200, 78712-1202
AUSTIN, TX, UNITED STATES OF AMERICA
E-mail address: stinga@math.utexas.edu